LITERATURE CITED

1. S. M. Kapustyanskii and V. N. Nikolaevskii, "Quantitative formulation of an elasto-plastic dilatation model (with sandstone as an example)," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 4 (1984).
2. A. N. Stavrogin and A. G. Protosenya, Plasticity of Rocks [in Russian], Nedra, Moscow (1979).
3. A. N. Stavrogin, B. G. Tarasov, et al., "Strength and deformation of rocks in the pre- and post-critical regions," Fiz.-Tekh. Probl. Razrab. Polezn. Iskop., No. 6 (1981).
4. P. E. Senseny, A. F. Fossum, and T. W. Pfeile, "Nonassociative constitutive laws for low porosity rocks," Int. J. Numer. Anal. Math, 7, 101 (1983).
5. W. Wawersik and Ch. Fairhurst, "A study of brittle fracture in laboratory compression experiments," Int. J. Rock Mech., 7, 561 (1970).
6. D. R. Stephens, E. M. Lilley, and H. Louis, "Pressure—volume equation of state of consolidated and fractured rocks to 40 kb," Int. J. Rock Mech., 7, 257 (1969).
7. V. N. Nilolaevskii, L. D. Livshitz, and I. A. Sizov, "Mechanical properties of rocks. Deformation and fracture," Itogi Nauki Tekhniki Ser. Mekh. Deformiruemogo Tverd. Tela, 11, VINITI, Moscow (1978).
8. A. N. Stavrogin, B. G. Tarasov, and E. D. Pevzner, "Effect of strain rate on the post-critical characteristics of rocks," Fiz.-Tekh. Probl. Razrab. Polezn. Iskop., No. 5 (1982).
9. E. Hoek, "Strength of jointed rock masses," Geotechnique, 33, No. 3 (1983).
10. N. Brook, "Estimating the triaxial strength of rocks," Int. J. Rock Mech., 16, 261 (1979).

NONAXISYMMETRIC SOLUTION BIFURCATION AND THE STABILITY OF SHELLS OF REVOLUTION WITH A SINGULAR PERTURBATION

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We are examining the phenomenon of branching of the modes of equilibrium of a thin shell without buckling. Engineers are often faced with the problem of choosing a shell structure from the desired stable state when it has several equilibrium positions. In engineering practice, such requirements are typical, for example, in the use of shells as protective exploding membranes, in pneumatic automation systems containing shell elements, etc. The difficulties encountered in analyzing this type of problem are directly related to one of the central problems in the membrane theory of shells — the existence of many stable modes for one value of the load parameter.

Here we analyze the post-critical deformation of nearly perfect thin elastic shells under the influence of pressure

$$\rho = q\eta + p,\ p^* = \min_n \{p_n\},$$

where q is a small scalar parameter; $\eta(\alpha, \beta)$ is a function characterizing the distribution of the perturbing pressure over the surface of the shell; $\{p_n\}$ are eigenvalues. The dependence of the branching of a nonaxisymmetric mode of loss of stability of a conical shell on the form of the η-function was established. It was found that nonaxisymmetric bifurcation is accompanied either by an explosion or by buckling. The phenomenon of buckling is characterized by the fact that attainment of the bifurcation point does not exhaust the load-carrying capacity of the shell. If it is energetically favorable, the nonaxisymmetric mode is seen in the static state [1].

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The conclusion regarding the stability of the solution is generally based on analysis of the Liapunov indices. One thereby establishes the stability or instability of the solution relative to perturbations of the initial data. Here we study features of the deformation of shells with different perturbations of pressure over the shell surface. Such an approach not only makes sense from a physical standpoint, but it is the approach that must be taken in a number of cases. The necessity of its use stems on the one hand from the fact that it is very complicated to monitor the distribution of pressure perturbations over the shell surface in the setup of an experiment and, on the other hand, from the fact that qualitative studies of the effect of these perturbations significantly refine the mechanism of loss of shell stability.

We will use U_+ to designate the set of nonaxisymmetric modes of loss of stability for which branching is accompanied by an explosion. We will use U_- to designate the set of modes the bifurcation of which corresponds to the mechanical phenomenon of buckling. A numerical experiment was conducted to determine laws in the distribution of elements of U_+ and U_- on the loading curve with the condition $\eta(r) \equiv 1$. It turned out that at high pressures, small nonaxisymmetric modes of a conical shell belong to one of four types of solutions differing in the method of loss of stability and the character of the dependence of the number of waves on the shell surface on the magnitude of the pressure.

It must be remembered that repeated attempts have been made previously to find post-critical modes of equilibrium. However, none led to a positive result in the class of axisymmetric solutions [2]. Here, we establish this property for solutions in the nonlinear theory of shells by virtue of two circumstances:

1. The post-critical analysis was performed not only in the axisymmetric formulation, but also in regard to post-bifurcational nonaxisymmetric modes.

2. Stable equilibrium modes branch appreciably only when the values of the geometric parameter of the shell correspond to the condition of singular perturbation by a single small parameter with a higher derivative. However, in this case a large number of nonaxisymmetric modes generally belonging to the U_+ set also branch [3]. Thus, for effective analysis, a method was developed to calculate all local modes of equilibrium on a specified segment of the loading curve and a computer program was written which automatically gives stable solutions.

1. Vibrational-Asymptotic Analysis of Nearly Perfect Shells. We will examine the stability of a thin elastic shell of revolution within the framework of geometrically nonlinear theory. The deformation of the shell contains nonlinear terms in a "quadratic" approximation

$$\varepsilon = L_1(V) + (1/2)L_2(V) + L_{11}(z, V), \tag{1.1}$$

where L_1, L_2, and L_{11} are linear, quadratic, and bilinear operators; V is the displacement vector function; z is a small perturbation of the middle surface of the shell.

Let the shell be deformed by a pressure $\rho(r) = p + q\eta(r)$, where p is a uniform external pressure equal to one of the eigenvalues $\{p_n\}$ of the corresponding nonlinear boundary-value problem linearized in the neighborhood of the axisymmetric solution; $\eta(r)$ is a positive, sufficiently smooth function of the radius r; q is a small scalar parameter; z is a perturbation of the middle surface of the shell which is close to a nonaxisymmetric mode of loss of stability V_1, i.e., $z = \tau V_1$.

We will assume that given geometric imperfections τ of sufficiently small amplitude, the displacement $V(\alpha, \xi, \rho)$ can be expanded into an asymptotic series in integral powers of the parameter ξ. The parameter ξ characterizes the additional displacements caused by branching of the nonaxisymmetric mode of loss of stability [1]:

$$V(\alpha, \xi, \rho) = V_0(r, \rho) + \sum_{k=1}^{\infty} \xi^k V_k(\alpha, \rho), \quad \{r, \varphi\} = \alpha. \tag{1.2}$$

Here, $V_0(r, \rho)$ is the vector of the axisymmetric displacements of the main stress–strain state.

We will study modes of loss of stability in a nonaxisymmetric formulation, using the principle of possible displacements. To do this, we insert (1.2) into (1.1) to obtain

$$\varepsilon = \tau L_{11}(V_1, V_0) + \sum_{k=0}^{\infty} \xi^k \varepsilon_k,$$
$$\varepsilon_k = L_1(V_k) + \frac{1}{2}\sum_{i=0}^{k} L_{11}(V_{k-i}, V_i), \quad k = 0, 1, \ldots \tag{1.3}$$

The analogous series for the virtual strain corresponding to the virtual displacement δV has the form

$$\delta\varepsilon = \tau L_{11}(V_1, \delta V) + \delta\varepsilon_0 + \sum_{k=1}^{\infty} \xi^k \delta\varepsilon_k, \tag{1.4}$$
$$\delta\varepsilon_0 = L_1(\delta V) + L_{11}(V_0, \delta V), \ \delta\varepsilon_k = L_{11}(V_k, \delta V).$$

It follows from power series (1.3) that the generalized stress σ can also be expanded into a series in integral powers of ξ. Meanwhile, the coefficients of this expansion σ_i satisfy the determining relation $\sigma_i = \Gamma(\varepsilon_i)$, $i = 0, 1, \ldots$. Considering this, we perform a variational analysis of the nonaxisymmetric mode of equlibrium in asymptotic form. We first note the following: the coefficient with ξ^0 in the expression for variation of the energy functional is identically zero, since the corresponding boundary-value problem describing the main axisymmetric stress—strain state was obtained on the basis of the variational principle; if $\rho = p$, the coefficient H_1 with ξ is also equal to zero since the expression for it is an eigenvalue problem in variations.

Having set $\delta V = V_k$ in H_1, we arrive at the identity

$$\int_S \{\sigma_{0,p} L_{11}(V_1, V_k) + \sigma_1 [L_1(V_k) + L_{11}(V_{0,p}, V_k)]\}\, ds = 0, \quad k = 1, 2 \ldots, \tag{1.5}$$

which satisfies the eigenfunction V_1 and the k-th Koiter approximtion. Here, S is the middle surface; the subscript p means that the quantity was calculated at $\rho = p$.

We will limit ourselves to displacements in the class of eigenfunctions. Then, with allowance for (1.3)-(1.5) and the Betti reciprocity theorem, we arrive at the asymptote form of the variational principle:

$$\sum_{k=1}^{\infty} \xi^k H_k = -\tau \int_S [\sigma_0 L_2(V_1) + \sigma_1 L_{11}(V_1, V_0)]\, ds + \ldots, \tag{1.6}$$
$$H_1 = \int_S [\sigma_0 L_2(V_1) + \sigma_1 \varepsilon_1]\, ds, \quad H_2 = \frac{3}{2}\int_S \sigma_1 L_2(V_1)\, ds,$$
$$H_3 = \int_S [2\sigma_1 L_{11}(V_1, V_2) + \sigma_2 L_2(V_1)]\, ds,$$
$$H_k = \int_S \left\{\sum_{i=2}^{k-1} \sigma_i L_{11}(V_1, V_{k-i}) + \sigma_1\left[\frac{1}{2}\sum_{i=1}^{k-1} L_{11}(V_{k-i}, V_i) + L_{11}(V_1, V_{k-1})\right]\right\} ds$$

(the terms of the form $O(\xi^m, \tau^k)$, $k \geq 1$, $m + k \geq 2$ have been discarded).

Let $0 < \tau < \tau_0$, $\tau_0 \ll 1$. We will use ξ_m to designate the value of ξ at which the function $q(\xi)$ reaches an extremum. We naturally assume that $\xi_m(\tau)$ and $q(\tau) \equiv q(\xi_m(\tau))$ — are sufficiently smooth functions of the parameter τ, while $\lim_{\tau\to 0} \xi_m(\tau) = 0$, $\lim_{\tau\to 0} q(\tau) = 0$. In the plane (q, τ), the function $q(\tau)$ is a line of extreme values. The sensitivity of shells to geometric imperfections can be determined from the location of the line $q(\tau)$ in the plane. If the curve $q(\tau)$ lies in the top half-plane, i.e., if $q(\tau) > 0$, $\tau \in (0, \tau_0)$, then we will assume that the shell is insensitive to small geometric imperfections. If the curve $q(\tau)$ is located in the lower half-plane, we place the shell in the class of shells sensitive to small geometric imperfections. It should be noted that the dominant terms of the asymptote of the coordinates of the point $(\xi_m(\tau), q(\xi_m)$ satisfy the necessary condition for existence of the extremum:

$$\sum_{k=1}^{\infty} k\xi_m^{k-1} H_k = O(\xi_m^{i-1}, \tau^l), \quad i, l \geqslant 1 \tag{1.7}$$

Having set $\xi = \xi_m$ in (1.6), we construct the solution of system (1.6-1.7) in the form of series

$$q = \sum_{k=1}^{\infty} q_{k\tau}\xi_m^k, \quad \tau = \sum_{k=1}^{\infty} \tau_k \xi_m^k, \tag{1.8}$$

where $q_{k\tau}$ and τ_k are unknown coefficients.

We expand $\sigma_0(\rho)$, $\varepsilon_0(\rho)$, and $V_0(\rho)$ into Taylor series in the neighborhood of the point $\rho = p$. We insert them into (1.7) and simplify the resulting expressions by means of (1.5). From here, we use the method of undetermined coefficients to obtain

$$q_{2\tau} = -3H_3/J_\eta, \quad J_\eta = \int_S \left[\sigma'_{0,p} L_2(V_1) + 2\sigma_1 L_{11}(\eta V'_{0,p}, V_1)\right] ds, \tag{1.9}$$

if $J_\eta \neq 0$ and $q_{1\tau} \equiv -2H_2/J_n = 0$.

Numerical analysis of the equations of spherical and conical shells shows (see Part 3, for example) that there is a discrete set of values of the geometric parameters λ for which the equality $H_3 = 0$ is valid at certain bifurcation points. Let $H_i = 0$ for $i = 2, 3, \ldots, k-1$ and $H_k \neq 0$. Then the first nontrivial coefficient has the form $q_{k-1,\tau} = -kJ_\eta^{-1}H_k$. We obtain the following from comparison of the Koiter parameters $q_{k-1,\tau}$ for an imperfect shell with the corresponding parameter q_{k-1} for an ideal shell [1, 4]

$$q_{k,\tau} = (k+1)q_k, \; k = 1, 2, \ldots \tag{1.10}$$

Thus, if $q_k > 0$, then $q_{k,\tau} > 0$. If $q_k < 0$, then $q_{k,\tau} < 0$. Consequently, the way in which an imperfect shell becomes unstable is determined by the sign of the Koiter parameter for a perfect shell with boundary conditions, initial form of the middle surface, geometric dimensions, and magnitude and distribution of pressure which coincide with the analogous quantities for a shell having small geometric imperfections.

We insert (1.8) into (1.6) and assume that $\xi = \xi_m$ at the extremum point of the curve $q(\xi)$. Equating the terms with identical powers of ξ_m, we use (1.5) and (1.10) to obtain

$$q^d \sim d \sqrt[k-1]{kq_{k-1}}\, \tau J_\eta^{-1} \int_S \sigma_1 L_1(V_1)\, ds, \quad d = \frac{k}{k-1}, \quad k = 3, 5 \ldots \tag{1.11}$$

Thus, the location of the curve $q(\tau)$ in the plane (q, τ) is determined by the function sign (q_{k-1}). It was shown in [1] that the sign of the Koiter parameters is determined by the sign of the functional

$$J_\eta = \int_0^1 u_c(r) \left[\int_0^r \eta(t)\, t\, dt\right] dr, \tag{1.12}$$

where $u_c(r)$ is the second approximation in the Koiter theory for the axisymmetric component of the normal displacement. It follows from this that for an oscillating function $u_c(r)$, the sign of the functional J_η and, thus, the sensitivity of the shell to small geometric imperfections depends on the η-function. The latter is determined by the conditions of the experiment: the design features of the experimental unit, the method of loading, etc.

The completed study makes it possible to distinguish a class of imperfect shells which are sensitive to pressure perturbations. Let a certain shell have an oscillating function $u_c(r)$ at the point $p \in \{p_n\}$. Then in accordance with (1.12), (1.9), and (1.8), by changing the distribution of the perturbing pressure on the shell surface we can change the sign of the parameter $q_{2,\tau}$ and, thus, the neighborhood in which the small mode of equilibrium exists. Since there is a fully determined correspondence between the mechanical phenomenon of loss of stability of a shell under a conservative load and the property of bifurcation of the solution, then we can place shells for which bifurcation of equilibrium is dependent on the η-function in the class of shells sensitive to external pressure perturbations. We should note that Eq. (1.11) was obtained in [4, 5] with k = 3 and $\eta(r) = 1$.

2. Initial Post-Critical Deformation of Shells of Revolution. We will use $\mu = h/a\gamma$ to designate the relative thickness of the shell, where a is the radius in plan, $\gamma^2 = 12(1 - \nu^2)$, ν is the Poisson's ratio, and h is the shell thickness. If we make the dimensionless transition by means of the formulas $\Phi_R = E\mu\Phi/a$, $w_R = aw$, $\rho_R = E\rho\mu^2/\gamma$, $r_R = ar$ (E is the Young's modulus), then the equations of the finite displacements in mixed form are

$$\begin{Vmatrix} \mu\Delta^2 & -\Delta_r \\ \Delta_r & \mu\Delta^2 \end{Vmatrix} \begin{Vmatrix} w \\ \Phi \end{Vmatrix} = \begin{Vmatrix} L(w,\Phi) \\ -\frac{1}{2}L(w,w) \end{Vmatrix} + \rho \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \tag{2.1}$$

$$L(w,\Phi) = w''H\Phi + \Phi''Hw - 2IwI\Phi,\ H() = r^{-1}[()' + r^{-1}()^{\cdot\cdot}],$$

$$I() = r^{-1}[()^{\cdot\prime} r^{-1} - ()^{\cdot\cdot}],\ ()' = \partial/\partial r,\ ()^{\cdot} = \partial/\partial\varphi,$$

a) $r \in \partial\Omega,\ w = w'' + \nu Hw = 0,\ H\Phi = -I\Phi = 0;$

b) $r \in \partial\Omega,\ w = w' = 0,\ H\Phi = -I\Phi = 0;$

c) $r \in \partial\Omega,\ w = w' = 0,\ \Phi'' - \nu H\Phi = 0,\ \Phi''' - H\Phi + 2(1+\nu)(\Phi^{\cdot\cdot\prime} + \Phi^{\cdot\cdot}) + \nu(\Phi - \Phi'' + \Phi' - \Phi^{\cdot\cdot\prime} + 2\Phi^{\cdot\cdot}) = 0,$

where $\Delta_r(\ ) = r^{-1}\{[raR_\varphi^{-1}(\ )']' + a(R_r r)^{-1}(\ )^{\cdot\cdot}\}$; R_φ and R_r are the principal radii of curvature of the middle surface of the shell; (φ, r) are polar coordinates. Equations (2.1) describe the formation of an elastic shallow shell with a middle plane which is identical with the coordinate plane. Since the Kirchhoff–Love hypotheses were used in the derivation of Eqs. (2.1), we assume that $\mu \ll 1$.

We will use the Poincaré–Liapunov method [6, 7] to analyze nonaxisymmetric bifurcation from the axisymmetric state $(w_0(r), \Phi_0(r))$. To do this, Eq. (2.1) and the boundary conditions are written in displacements. Their linearization in the neighborhood of zero yields an eigenvalue problem and its conjugate problem. Comparison of these problems establishes the relationship between the eigenfunctions of the problems. Using these functions, we regularize the perturbation problem. After some transformations are performed, the solution of the perturbation problem is approximated by a segment of a Poincaré–Liapunov series

$$X \sim \sum_{i=1}^{d}\sum_{j=1}^{k} X_{ij}\xi^i q^j,\quad X = (w - w_0,\ \Phi - \Phi_0),$$

which reduces the problem to a recurring sequence of boundary-value problems:

the nonlinear problem of axisymmetric deformation of the shell

$$\mu Au - r\theta v = r^{-1}\left(uv + \int_0^r p_n t\,dt\right),\ A(\ ) \equiv \left[r^{-1}\frac{d}{dr}r(\ )\right]', \tag{2.2}$$

$$\mu Av + r\theta u = -\frac{1}{2}r^{-1}u^2,\quad \theta = a/R_\varphi,\quad \{u, v\} = \{w, \Phi\}'_0,$$

$$r = 0, 1,\ u' + \nu u = v = 0,\ u = v = 0,\ u = v' - \nu v = 0;$$

the eigenvalue problem

$$\begin{Vmatrix} \mu\Delta^2 + r^{-1}v(\ )'' + v'H & -\Delta_r - u'H - r^{-1}u(\ )'' \\ \Delta_r + u'H + r^{-1}u(\ )'' & \mu\Delta^2 \end{Vmatrix} \begin{Vmatrix} w_n \\ \Phi_n \end{Vmatrix} = 0, \tag{2.3}$$

$$\lambda_i^{(1)} w_n = \lambda_i^{(2)}\Phi_n = 0,\quad i = 1, 2,\quad r \in \partial\Omega;$$

problems of post-critical deformation

$$\left[B + \begin{Vmatrix} w_n S_w & w_n S_\Phi \\ -\Phi_n S_\Phi & -\Phi_n S_\Phi \end{Vmatrix}\right] \begin{Vmatrix} w_{20} \\ \Phi_{20} \end{Vmatrix} = \begin{Vmatrix} L(\Phi_n, w_n) \\ -\frac{1}{2}L(w_n, w_n) \end{Vmatrix}, \tag{2.4}$$

$$\lambda_i^{(1)}w_{20}=\lambda_i^{(2)}\Phi_{20}=0,\quad i=1,2,\ r\in\partial\Omega,\quad S_w(\ )=\int_0^1\int_0^{2\pi} rw_n(\ )\,dr\,d\varphi,\quad w\rightleftarrows\Phi;$$

$$\left[B+\begin{Vmatrix} w_nS_w & w_nS_\Phi \\ -\Phi_nS_\Phi & -\Phi_nS_w\end{Vmatrix}\right]\begin{Vmatrix} w_{01} \\ \Phi_{01}\end{Vmatrix}=q\begin{Vmatrix}\eta \\ 0\end{Vmatrix}, \tag{2.5}$$
$$\lambda_i^{(1)}w_{01}=\lambda_i^{(2)}\Phi_{01}=0,\quad i=1,2,\quad r\in\partial\Omega$$

(B is a 2 × 2 matrix from (2.3)).

We expand the function $\eta(r, \varphi)$ into a Fourier series. We introduce notation $q = q_n\eta_n(r)$, where $\eta_n(r)$ are coefficients of the expansion. Using the Fredholm alternative and the explicit properties of boundary-value problems (2.2)-(2.5) for the coefficients X_{ij}, it can be shown that the approximate branching equation has the form

$$L_{01}q_n+L_{11}\xi q_0+L_{30}\xi^3+\ldots=0, \tag{2.6}$$
$$L_{01}=\frac{1}{2}\int_0^1\eta_n(t)\,\omega_n(t)\,t\,dt,\quad L_{11}=-\int_0^1 u_c(t)\left[\int_0^t\eta_0(r)\,r\,dr\right]dt,$$
$$L_{30}=\int_0^1\left[v_cg_2+u_cg_1+\frac{1}{2}r(\omega_{2n}h_1-h_{2n}h_2)\right]r\,dr;$$

$$\{w_{20},\Phi_{20}\}=\{\omega_{2n}(r), f_{2n}(r)\}\cos 2(n\varphi+\alpha_n)+\left\{\int_1^r u_c(t)\,dt,\ \int_0^r v_c(t)\,dt\right\}, \tag{2.7}$$
$$\{w_n,\Phi_n\}=\{\omega_n(r), f_n(r)\}\sin(n\varphi+\alpha_n),$$

where $\{\omega_{2n}(r), f_{2n}(r)\}$, $\{u_c(r), v_c(r)\}$ are vector functions of the solution of the inhomogeneous boundary-value problems obtained (2.4) by means of (2.7); g_1, g_2, h_1, h_2 are the inhomogeneous parts of the corresponding boundary-value problems; α_n is the initial phase for nonaxisymmetric harmonics. It follows from the branching equation that if $q_n \neq 0$, $q_0 \neq 0$, then the number of branched modes of equilibrium depends on the ratio of the parameters q_n and q_0. This result was obtained by applying Newton's diagramming technique to Eq. (2.6) [8].

Let the perturbing pressure be axisymmetric and $L_{30} \neq 0$, $L_{11} \neq 0$. Then one nonaxisymmetric mode having a positive amplitude ξ branches off in the neighborhood of the critical pressure. The mechanical quantities for this mode (diplacements, generalized stresses, strains) can be represented in the form of series in powers of q_0 which are multiples of 1/2. If $\mathrm{sign}(L_{30}L_{11}) < 0$, then the load-carrying capacity of the shell is not exhausted upon attainment of the critical pressure, while the nonaxisymmetric mode is observed experimentally when it is energetically favorable. The process of loss of stability occurs in the form of buckling. If $\mathrm{sign}(L_{30}L_{11}) > 0$, then bifurcation is accompained by an explosion. Another Poincaré—Liapunov expansion occurs with the following limitations on the branching equation: $L_{30} = 0$, $L_{11} \neq 0$, $L_{50} \neq 0$ (the coefficient L_{40} is equal to zero). In this case, the mechanical quantities can be represented in the form of a series in powers of q_0 which are multiples of 1/4. Buckling of the shell occurs as before, but it is necessary to replace L_{30} by L_{50}.

Let $L_{11} \neq 0$, $L_{30} \neq 0$, $L_{01} \neq 0$. Then we have the equation of the curve $q_n = 2L_{11}q_0\sqrt{q_0/3g}/3L_{01}$, $g = -L_{30}/L_{11}$ dividing the plane (q_0, q_n) into regions each of which contains a constant number of equilibrium modes. We will use q_n^+ and q_n^- to designate the branches of this curve. Using Newton's diagramming technique for $g > 0$, we can show that there are three nonaxisymmetric modes in the right semicircle of the coordinate origin between q_n^+ and q_n^-. One equilibrium mode branches off in the regions above q_n^+ and below q_n^-. If $q_n > q_n^+$, then $\xi > 0$ when $L_{11}L_{01} > 0$. If $q_n < q_n^-$, then $\xi > 0$ when $L_{11}L_{01} < 0$. For $g < 0$, then the above-described regions in the plane (q_0, q_n) will be mirror reflections relative to the q_0 axis.

It follows from the above that the number of small nonaxisymmetric modes depends on the sign of the Koiter parameter and the sign of the quantity $(L_{11}L_{01})$. Thus, the shell will be sensitive to the distribution of the perturbing pressure if u_c or ω_n are oscillating functions.

We will use ℓ to designate dim(Ker B) — the dimension of the null subspace of the operator B at the point $p = p_n$. If $\ell > 1$, then to analyze the equilibrium modes it is necessary to use the multidimensional variant of the Poincaré–Liapunov method. We set $\ell^+ = \max_{\lambda,p}$ It is established below that if $p > p^*$, where $p^* = \min_n\{p_n\}$, but it is less than a certain p^+, then $\ell^+ = 2$. If $p^+ \leq p < p^-$, then $\ell^+ = 4$. In the last case, the dimension ℓ may take values of 0, 1, 3, and 4. We should point out several general laws governing the change in dim(Ker B). The value of ℓ^+ does not decrease with an increase in p. The value of ℓ^+ can only change by two. Finally, the value of ℓ^+ may increase as μ approaches zero.

3. Initial Post-Critical Deformation of a Conical Shell. Shown below are results of numerical integration of the equations of a conical shell (2.1c) loaded by a uniform external pressure. The shell material has a Poisson's ration $\nu = 0.34$.

Figure 1 shows the development of $u_c(r)$ at $\lambda = 14.22$ and different n. The numbers of the curves correspond to the eigenvalues $\{p_n\}$. Henceforth, the shell geometry is characterized by the parameter λ, equal to $\sqrt{\theta/\mu}$, where θ is the half-opening of the shell.

It is apparent that $u_c(r)$ is an oscillating function. Thus, nonaxisymmetric bifurcation of the mode of equilibrium of a thin conical shell and, thus, the mechanism of loss of stability depend on the perturbing pressure in the neighborhood of the spectrum points. Moreover, according to (1.11), the sensitivity of this shell to geometric imperfections depends on the η-function. This conclusion is sufficiently general in character, since it is also valid for a spherical shell [1]. The same conclusion can be reached for a cylindrical shell with a fixed end and under compression in the axial direction. This follows from Eq. (1.11) and the numerical results in [9]. The data in Fig. 1 also indicates that to determine the mechanism of loss of stability (whether or not the shell will explode), it is necessary to specify a corresponding perturbation η for each n. Meanwhile, in problems such as these, it is important primarily to monitor pressure perturbations near the edge. This effect is particularly strong for a thin conical shell, with the nonaxisymmetric mode being characterized by a large number of waves (see the curves with n = 10, 14, for example).

Figure 2 shows the integral function $n(p_n)$ for $\lambda = 14.22$, where n is equal to the number of waves in the circumferential direction for a small nonaxisymmetric mode. Here, point 1 corresponds to nonaxisymmetric modes for which bifurcation under uniform external loading is accompanied by an explosion, while point 2 corresponds to buckling of the conical shell.

Analysis of the mechanical phenomena corresponding to branching of the solution was accomplished by the following scheme. We numerically determined the values of the functions of branching equation (2.6). Using the Newton diagram and the Poiseuille theorem, we established the neighborhood of existence of the minor mode of equilibrium ξ_n and studied the potential energy Π_n in the neighborhood of ξ_n. If $p > p_n$ and the second variation

$$\delta^2\Pi_n \equiv -4\xi_n^2 L_{30} \tag{3.1}$$

is greater than zero, then bifurcation of the solution corresponds to buckling. If the inequality $\delta^2\Pi_n < 0$ is valid at $p < p_n$, the bifurcation of the solution is accompanied by explosion of the shell. Having applied the Hamilton–Ostrogod principle in the Landau approximation [10], we reduce the dynamical equation for ξ_n to the form

$$\frac{d^2\xi_n}{dt^2} = \sigma L_{11}\xi_n + L_{30}\xi_n^3, \quad \sigma = p - p_n.$$

It can be determined from this that branching of the shell occurs in an oscillatory regime in the first case. In the case, there is severe loss of stability, and the nonaxisymmetric perturbation increases exponentially during the loss of stability.

It is evident from Fig. 2 that $p_7 = \min_n\{p_n\}$ is a simple point of the spectrum. At $0.392 < p < 0.804$, the function $n(p_n)$ has two branches. These branches describe rapidly and slowly oscillating waves. At $p > 0.804$, the function has four branches. Two of the branches increase with an increase in pressure, while two decrease. It is for this reason that two types of loss of stability, when $p \in [0.804; 1.44]$, are possible over the loading curve w(p). If

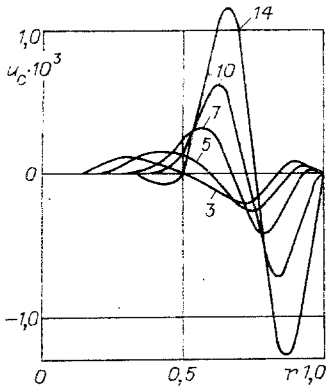

Fig. 1

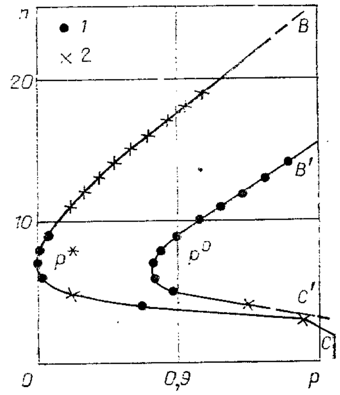

Fig. 2

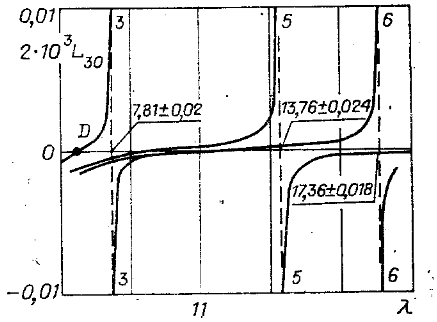

Fig. 3

the spectrum point belongs to the branches Bp^*, then buckling occurs. If the spectrum point belongs to the branches $B'p^0$, then an explosion occurs. Thus, alternation of the points of the spectrum on the curve w(p) which belong to the branches Bp^* and $B'p^0$ leads to a large variety of nonaxisymmetrical modes of loss of stability. For the shell being examined, the pressure p = 0.804 is doubly degenerate. Thus, in the neighborhood of this point, the initial post-critical deformation is very sensitive to geometric imperfections [11].

Numerical integration of the equations of a conical shell in a broad range of λ revealed several laws governing nonaxisymmetric modes of loss of stability. When certain λ and p the function $n(p_n)$ acquires "new" branches (such as in the neighborhood of the point p^0), then an especially large number of nonaxisymmetric modes with a negative Koiter parameter g appear on the loading curve. These shells are distinguished by the fact that the presence of geometric imperfections having the same rotation group as the corresponding eigenfunction reduce the value of the p coordinate of the bifurcation point. In contrast to this, the bifurcation points (points 2 in Fig. 2) are shifted to the right in the presence of such geometric imperfections. Detailed analysis of each bifurcation point by the Poincaré—Liapunov method showed that at λ = 14.22, 35 modes having a positive amplitude ξ branch off of the axisymmetric mode if the perturbing pressure is axisymmetric.

A distinctive feature of the nonaxisymmetric bifurcation of a conical shell with large λ is that the range of pressure in which only the axisymmetric mode exists is significantly smaller than the range of pressure in which nonaxisymmetric equilibrium modes exist.

There are several exceptions in nonaxisymmetric bifurcation. There is no regularity in the alternation of spectrum points on the branch p^*c. Figure 3 shows graphs for L_{30}. The numbers of the curves correspond to the eigenvalues $\{p_n\}$. It is apparent that for one λ the functional L_{30} can take different values with a change in n and, in accordance with (3.1), both the sign of $\delta^2\Pi_n$ and the mechanism of loss of stability change. In particular, at

$\lambda = 14.22$, $L_{30} < 0$ for n = 5 and $L_{30} > 0$ for n = 6. This explains the change in the character of bifurcation in the transition from n = 5 to n = 6 on the branch p^*c (see Fig. 2). It should be noted that L_{30} changes sign either in the neighborhood of points of continuity (such as in the neighborhood of the point D) or in the neighborhood of second-order points of discontinuity. In the first case, in order to study the initial post-critical deformation of a thin shell, it is necessary to retain terms of the order $O(\xi^{\alpha}), \alpha > 3$, in Eq. (2.6). In the second case, it is usually necessary to change the algorithm for numerical integration of the boundary-value problems, approximating the solution with Chebyshev polynomials [12]. For the problem being discussed, this property is manifest not only in the eigenvalue problem — as occurs for the Navier—Stokes equations — but also in analysis of the leading coefficients of the branching equation. Points of the second type were seen in [5] in the local loading of shells.

In conclusion, we should note that no qualitatively new features in the mechanism of loss of stability were discovered from study of the post-critical deformation of shells with hinged-stationary and sliding-fixed bearing contours.

LITERATURE CITED

1. V. V. Larchenko, "Asymptotic analysis of nonaxisymmetric modes of equilibrium of a thin shallow spherical shell," Prikl. Mat. Mekh., 44, No. 6 (1980).
2. D. I. Shil'krut and P. M. Vyrlan, "Stability of geometrically nonlinear shells," Dokl. Akad. Nauk SSSR, 225, No. 4 (1975).
3. V. M. Kornev and V. M. Ermolenko, "Sensibility of shells to buckling disturbances in connection with parameters of critical loading spectrum," Int. J. Eng. Sci., 18, 379 (1980).
4. W. T. Koiter, "The stability of elastic equilibrium," Air Force Flight Dynamics Lab. Tech. Rept. AFFDL-TR-70-25 (1970).
5. J. R. Ritch, "The buckling and post-buckling behavior of spherical caps under concentrated load," Int. J. Solids Struct., 4 (1968).
6. H. Poincaré, "Sur le problems de trois corps et les equations de la dynamique," Acta Math., 13 (1980).
7. A. M. Liapunov, "On one Chebyshev problem," in: Collected Works [in Russian], Vol. 3, Izd. Akad. Nauk SSSR, Moscow (1959).
8. N. G. Chebotarev, "Newton polygon and its role in the current development of mathematics," in: Collected Works, Vol. 3, Izd. Akad. Nauk SSSR, Moscow (1950).
9. N. Yamaki, "Post-critical behavior and flaw sensitivity of a circular cylindrical shell loaded in compression," in: Theoretical and Applied Mechanics: Transactions of the XIV International Conference IUTAM [Russian translation], Mir, Moscow (1979).
10. L. D. Landau, "On the problem of turbulence," Dokl. Akad. Nauk SSSR, 44, No. 8 (1944).
11. E. Byskov and J. M. Hutchinson, "Mode interaction in axialy stiffened cylindrical shells," AIAA J., 15, No. 7 (1977).
12. K. I. Babenko and M. M. Vasil'ev, "Verifying calculations in a problem on the stability of a plane Poiseuille flow," Dokl. Akad. Nauk SSSR, 273, No. 6 (1983).